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APPROXIMATION OF OPTIMAL GAME STRATEGIES BY CONTINUOUS FUNCTIONS

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We consider three typical game problems in conflict-control systems. We establish that in the regular case the optimal methods of control can be approximated by continuous strategies so as to achieve an effect as near optimal as desired (from the viewpoint of the pursuer or the pursued).

1. Let us consider the motion $x(t) = \{x_i(t)\}$, $(i = 1, \ldots, n)$ described by the vector differential equation

$$dx / dt = A(t)x + B(t)u - C(t)v + f(t) \tag{1.1}$$

Here $A(t)$, $B(t)$, $C(t)$ are matrices of dimensions $n \times n$, $n \times r$, $n \times s$ respectively; $f(t)$ is an n-dimensional perturbation vector; u and v are control vectors of dimensions

r and s, constrained at each instant by the conditions

$$u[t] \in U_t, \qquad v[t] \in V_t \tag{1.2}$$

where U_t and V_t are closed bounded convex sets in the r- and s- dimensional spaces E_r and E_s, respectively, varying continuously with changing t.

We consider the following three problems (see [1–6].

1. The conflict encounter problem. The final instant ϑ is given. The problem of the pursuer, being guided by the control $u[t]$, is to minimize the quantity $\|\{x(\vartheta)\}_m\|$, while the pursued, being guided by the control $v[t]$, strives to maximize this quantity. Here the symbol $\{q\}_m$ denotes a vector composed from the first m coordinates of vector q, and $\|q\|$ denotes the Euclidean norm of vector q.

2. The pursuit problem. We are given a closed bounded convex set M in an m-dimensional space X_m. The pursuer strives to bring the motion $\{x[t]\}_m$ onto the set M in the shortest possible time.

3. The evasion problem. Once again we are given a closed bounded convex set M in an m-dimensional space X_m. The pursued strives to keep $\{x[t]\}_m$ from hitting onto the set M. If he does not succeed in doing this, his aim becomes to postpone of bringing the motion $\{x[t]\}_m$ onto the set M as long as possible.

In all the problems we assume that the initial state $x[t_0] = x_0$ of system (1.1) is fixed.

In correspondence with [7], by the strategies $U(t, x)$ and $V(t, x)$ of the first and the second player we shall mean closed bounded convex sets, specified for each value of $\{t, x\}$, upper semicontinuous relative to inclusion, where $U(t, x) \subset U_t$ and $V(t, x) \subset V_t$, and by a solution of Eq. (1.1) we shall mean any absolutely continuous function $x(t)$ satisfying almost everywhere the condition

$$dx/dt \in A(t)x + B(t)U(t, x) - C(t)V(t, x) + f(t) \tag{1.3}$$

Then, in correspondence with the results in [4], the minimax strategy $U_1^\circ(t, x)$ in Problem 1 satisfies the following relation:

$$\begin{aligned}\Gamma^\circ(t_0, x_0) = \max_v \max_{x[t]} \{\|\{x(\vartheta)\}_m\| \mid \chi[U_1^\circ(t, x), V(t, x), t_0 x_0]\} \leqslant \\ \leqslant \max_v \max_{x[t]} \{\|\{x(\vartheta)\}_m\| \mid \chi[U(t, x), V(t, x), t_0, x_0]\}\end{aligned} \tag{1.4}$$

In Problem 2 the minimax strategy $U_2^\circ(t, x)$ satisfies the relation

$$\begin{aligned}\theta^\circ(t_0, x_0) = \max_v \max_{x[t]} \{\theta^\circ \mid \chi[U_2^\circ(t, x), V(t, x), t_0, x_0]\} \leqslant \\ \leqslant \max_v \max_{x[t]} \{\theta \mid \chi[U(t, x), V(t, x), t_0, x_0]\}\end{aligned} \tag{1.5}$$

Here and subsequently, $\chi[U(t, x), V(t, x), t_0, x_0]$ is a family of motions of system (1.3), generated by the strategies $U(t, x)$, $V(t, x)$, and by the initial data t_0, x_0, while θ is the time it takes for the motion $\{x[t]\}_m$ to go onto the set M, where

$$x[t] \in \chi[U(t, x), V(t, x), t_0, x_0].$$

The maximin strategies $V_1^\circ(t, x)$ and $V_2^\circ(t, x)$ in Problems 1 and 3 satisfy the conditions

$$\begin{aligned}\Gamma_0(t_0, x_0) = \min_u \min_{x[t]} \{\|\{x(\vartheta)\}_m\| \mid \chi[U(t, x), V_1^\circ(t, x), t_0, x_0]\} \geqslant \\ \geqslant \min_u \min_{x[t]} \{\|\{x(\vartheta)\}_m\| \mid \chi[U(t, x), V(t, x), t_0, x_0]\}\end{aligned} \tag{1.6}$$

$$\begin{aligned}\theta_0(t_0, x_0) = \min_u \min_{x[t]} \{\theta_0 \mid \chi[U(t, x), V_2^\circ(t, x), t_0, x_0]\} \geqslant \\ \geqslant \min_u \min_{x[t]} \{\theta \mid \chi[U(t, x), V(t, x), t_0, x_0]\}\end{aligned} \tag{1.7}$$

If the regular case holds in Problem 1 (for example, see [4, 7]), then a method is known for constructing the minimax and the maximin strategies of the pursuer and of the pursued .Here a case is called regular if the maximum of the expression

$$\varepsilon^\circ (t, x) = \max_{\|l\|=1} [\rho_2 (t, \vartheta, l) - \rho_1 (t, \vartheta, l) - l' x^\circ (t, x, \vartheta)] \tag{1.8}$$

for $\varepsilon^\circ (t, x) > 0$ is reached on a single vector among all vectors of the form

$$l = \{l_i\} \quad (i=1,\ldots,n), \qquad l_j = 0 \quad \text{for } j > m \tag{1.9}$$

The prime denotes transposition, while the meaning of the quantities occuring in (1, 8) is defined by the relations

$$\rho_1 (t, \vartheta, l) = \max_{u(\xi)} \Big[\int_t^\vartheta B' (\xi) S [\xi, t] lu (\xi) d\xi \Big], \quad u (\xi) \in U_\xi \tag{1.10}$$

$$\rho_2 (t, \vartheta, l) = \max_{v(\xi)} \Big[\int_t^\vartheta B' (\xi) S [\xi, t] lv (\xi) d\xi \Big], \quad v (\xi) \in V_\xi \tag{1.11}$$

$$x^\circ (t, x, \vartheta) = X [\vartheta, t] x + \int_t^\vartheta X [\vartheta, \xi] f (\xi) d\xi \tag{1.12}$$

where $X [t, \tau]$, $S [t, \tau]$ are the fundamental matrices of the equations

$$dx / dt = A (t)x, \ ds / dt \in - A' (t)s \tag{1.13}$$

Below we consider the approximations of the optimal strategies $U^\circ (t, x)$, $V^\circ (t, x)$ by continuous strategies, i.e., by those strategies which are given by single-valued continuous functions $u (t, x)$, $v (t, x)$, satisfying the conditions

$$u (t, x) \in U_t, \qquad v (t, x) \in V_t$$

We shall show in the following that such an approximation holds in the Problems 1–3 stated. The precise sense of this approximation will be made clear during the analysis of each of the three problems.

2. We consider Problem 1. Let us first dwell on the scalar case when $B (t) = b (t)$ is an n-dimensional vector and the set U_t is representable in the form

$$U_t = \{u : \ | u | \leqslant \mu, \ \mu > 0\} \tag{2.1}$$

and the regular case holds. Then the minimax strategy of the first player is determined as follows [4]: when $\varepsilon^\circ (t, x) > 0$ the set $U^\circ (t, x)$ is formed by those and only those quantities u°, for which the condition

$$s' (t, x) b (t) u^\circ = \max_u [s' (t, x) b (t) u] \ , \text{ for } | u | \leqslant \mu \tag{2.2}$$

is fulfilled, where $s (t, x)$ is the solution of Eq. (1, 2) under the boundary condition $s (\vartheta) = l^\circ$ and the vector l° is determined from (1, 8). In the region $\varepsilon^\circ (t, x) \leqslant 0$ we have $U^\circ (t, x) = U_t$. Note that the relations

$$\Gamma^\circ (t_0, x_0) = \Gamma_0 (t_0, x_0) = \varepsilon^\circ (t_0, x_0), \quad \text{if } \ \varepsilon^\circ (t_0, x_0) > 0 \tag{2.3}$$

$$\Gamma^\circ (t_0, x_0) = \Gamma_0 (t_0, x_0) = 0, \quad \text{if } \quad \varepsilon^\circ (t_0, x_0) \leqslant 0$$

are fulfilled in the regular case of the game.

The following theorem is valid under the assumptions made:

Theorem 2.1. For any $\alpha > 0$ there exists a continuous strategy $u^\alpha (t, x)$ satisfying the relation

$$\sup_v \sup_{x[t]} \{\|\{x(\vartheta)\}_m\| \mid \chi [u^\alpha(t, x), v(t), t_0, x_0]\} \leqslant \Gamma^\circ(t_0, x_0) + \alpha \tag{2.4}$$

where $v(t)$ is any admissible realization, i. e. an integrable function satisfying the condition $v(t) \in V_t$.

Let us prove the theorem. We introduce the function $\omega(t, x) = s'(t, x) b(t)$. Then, taking (2.1) into account, from (2.2) we get that the minimax strategy is determined by the conditions

$$U^\circ(t, x) = \begin{cases} \mu, & \text{if} \quad \omega(t, x) > 0 \ \text{ and } \ \varepsilon^\circ(t, x) > 0 \\ -\mu, & \text{if} \quad \omega(t, x) < 0 \ \text{ and } \ \varepsilon^\circ(t, x) > 0 \\ -\mu \leqslant u^\circ \leqslant \mu, & \text{if} \quad \omega(t, x) = 0 \quad \text{or} \quad \varepsilon^\circ(t, x) \leqslant 0 \end{cases} \tag{2.5}$$

We take an arbitrary number $\alpha > 0$ and in the $(n + 1)$-dimensional space $X_t\{\vartheta\} = \{t, x: t \leqslant \vartheta\}$ we define the sets $N_1, N_2, N_3, N_4, N_5, N_6$ in the following manner:

$$N_1 = \{t, x: \omega(t, x) > {}^1/_4\alpha\mu(\vartheta - t_0); \quad \varepsilon^\circ(t, x) > 0\} \quad N_4 = \{t, x: \varepsilon^\circ(t, x) > {}^1/_2\alpha\}$$
$$N_2 = \{t, x: \omega(t, x) < -{}^1/_4\alpha\mu(\vartheta - t_0); \ \varepsilon^\circ(t, x) > 0\} \quad N_5 = \{t, x: 0 < \varepsilon^\circ(t, x) \leqslant {}^1/_2\alpha\}$$
$$N_3 = \{t, x: |\omega(t, x)| < {}^1/_4\alpha\mu(\vartheta - t_0), \ \ \varepsilon^\circ(t, x > 0\} \quad N_6 = \{t, x: \varepsilon^\circ(t, x) \leqslant 0\}$$

Note that the sets N_4, N_5, N_6 do not intersect and comprise the whole space $X_t\{\vartheta\}$. In $X_t\{\vartheta\}$ we now define a strategy $u^\alpha(t, x)$ by setting

$$u^\alpha(t, x) = \begin{cases} u(t, x), \ \{t, x\} \in N_4, \\ u^*(t, x) = 2\alpha^{-1}\varepsilon^\circ(t, x) u(t, x), \ \{t, x\} \in N_5, \\ 0, \ \{t, x\} \in N_6, \end{cases}$$

$$u(t, x) = \begin{cases} \mu, \ \{t, x\} \in N_1 \\ -\mu, \ \{t, x\} \in N_2 \\ \mu^* = 4\mu^2\alpha^{-1}(\vartheta - t_0)\,\omega(t, x) \ \{t, x\} \in N_3 \end{cases} \tag{2.6}$$

Here $u(t, x)$ is a continuous function. Obviously, $u^\alpha(t, x)$ is defined and is continuous on the whole space $X_t\{\vartheta\}$.

Let us now see how $\varepsilon^\circ(t, x)$ from (1.8), will vary in the region N_4 when the second player uses any admissible realization $v(t)$. In the regular case the function $\varepsilon^\circ(t, x)$ is differentiable in the region $\varepsilon^\circ(t, x) > 0$ and, therefore, by computing the derivative $d\varepsilon^\circ/dt$ along a motion of system (1.1), generated by the controls $u^\alpha(t, x)$, $v(t)$, we obtain (for example, see [8])

$$d\varepsilon^\circ / dt = \max_u[\omega(t, x)u] - \omega(t, x)u^\alpha(t, x) - \max_v[s'(t, x)\,C(t)v] + s'(t, x)C(t)v(t)$$
$$\text{for } |u| \leqslant \mu, \ v \in V_t \tag{2.7}$$

and taking into account that in the region N_4 the continuous strategy $u^\alpha(t, x)$ differs from $U^\circ(t, x)$ from (2.5) and (2.6), only on the set N_3, we have

$$\frac{d\varepsilon^\circ}{dt} \leqslant \frac{\alpha}{2(\vartheta - t_0)} \tag{2.8}$$

From (2.8), (2.6) we obtain

$$\varepsilon^\circ[\vartheta] \leqslant \varepsilon^\circ[t_0] + \alpha$$

Then it follows from (1.8), (2.3), (2.6) that

$$\|\{x(\vartheta)\}_m\| = \varepsilon^\circ[\vartheta] \leqslant \Gamma^\circ(t_0, x_0) + \alpha \tag{2.9}$$

Hence follows the theorem's assertion.

By analogous calculations we can show that under a suitable approximation of the maximin strategy $V^\circ(t, x)$ by the continuous function $v^\alpha(t, x)$ in the regular case of the encounter problem and under constraints on the control of the form

$$V_t = \{v: |v| \leqslant \nu, \ \nu > 0\} \tag{2.10}$$

the following condition holds:

$$\inf_u \inf_{x[t]} \{\| \{x(\vartheta)\}_m \| \mid \chi[u(t), v^\alpha(t, x), t_0, x_0]\} \geqslant \Gamma_0(t_0, x_0) + \alpha \tag{2.11}$$

where $u(t)$ is an admissible realization. Thus, here too we can construct a continuous strategy $v^\alpha(t, x)$, approximating the maximin strategy such that the result of the game will differ from the optimal result by an amount as small as desired.

3. We now consider Problem 2 (on pursuit) stated in Sect. 1. We shall assume that the constraint on the pursuer's control is given in form (2.1) and that the regular case holds. Here the case is regular if the maximum of the expression

$$\varepsilon^\circ(t, x, \vartheta) = \max_{\|l\|=1} [\rho_2(t, \vartheta, l) - \rho_1(t, \vartheta, l) - l'x^\circ(t, x, \vartheta) - \max_p \{l'p\}] \tag{3.1}$$

for any $\vartheta \geqslant t_0$ and for the condition $\varepsilon^\circ(t, x, \vartheta) > 0$ is achieved on a single vector l° among the vectors of form (1.9). In relation (3.1) the quantities $\rho_1(t, \vartheta, l)$, $\rho_2(t, \vartheta, l)$, $x^\circ(t, x, \vartheta)$ are determined from (1.10)-(1.13), and the maximum of the quantity $\{l'p\}$ is taken over all vectors p such that $-p \in M$.

Then there exists (see [5]) a pursuer's strategy $U^\circ(t, x)$ which ensures that $\{x[t]\}_m$ is guided onto the set M at the absorption instant $\vartheta = \vartheta_0^M(t_0, x_0)$, where $\vartheta_0^M(t_0, x_0)$ is the smallest possible time in which the region of attainability of process (1.1), for all possible programed controls $u(t)$ and for any fixed programed control $v(t)$, will first contain at least one point of the set M.

The strategy $U^\circ(t, x)$ can be determined in the following way by using the results in [6]. In the $(n+1)$-dimensional space $X_t\{\vartheta_0^M(t_0, x_0)\}$ we introduce the sets W_1 and W_2:

$$W_1 = \{t, x: \varepsilon^\circ(t, x) > 0\}, \qquad W_2 = \{t, x: \varepsilon^\circ(t, x) \leqslant 0\}$$

Then, on the set W_1 the strategy $U^\circ(t, x)$ is determined by those and only those quantities u° for which the condition

$$s'(t, x) b(t) u^\circ = \max_u [s'(t, x) b(t) u] \qquad \text{for } |u| \leqslant \mu \tag{3.2}$$

is fulfilled; $U^\circ(t, x) = U_t$ on the set W_2. In formula (3.2) $s(t, x)$ is determined from conditions (1.8)-(1.13) and is a continuous function of all its arguments in the open region W_1.

The following theorem holds:

Theorem 3.1. For any $\alpha > 0$ there exists a continuous strategy $u^\alpha(t, x)$ such that the condition

$$\sup_v \sup_{x[t]} \min_\vartheta \{\rho[\{x(\vartheta)\}_m, M] \mid \chi[u^\alpha(t, x), v(t), t_0, x_0]\} \leqslant \alpha \tag{3.3}$$

$$\text{for} \quad t_0 \leqslant \vartheta \leqslant \vartheta_0^M(t_0, x_0)$$

is fulfilled at the instant $\vartheta_0^M(t_0, x_0)$. Here $\rho[q, M]$ is the Euclidean distance from the point q to the set M. The proof of Theorem 3.1 does not differ essentially from the proof of Theorem 2.1 and, therefore, we merely sketch the arguments.

In the $(n+1)$-dimensional space $X_t\{\vartheta_0^M(t_0, x_0)\}$ we define the sets N_1 and N_2

$$N_1 = \left\{t, x : |s'(t, x)\, b(t)| > \frac{\alpha}{4\mu\,(\vartheta_0^M(t_0, x_0) - t_0)} \text{ and } \varepsilon^\circ(t, x, \vartheta_0^M(t_0, x_0)) > \frac{\alpha}{2}\right\}$$
$$N_2 = \left\{t, x : |s'(t, x)\, b(t)| \leqslant \frac{\alpha}{4\mu\,(\vartheta_0^M(t_0, x_0) - t_0)} \quad \text{or} \quad \varepsilon^\circ(t, x, \vartheta_0^M(t_0, x_0)) \leqslant \frac{\alpha}{2}\right\}$$

Just as in Theorem 2.1, here we have constructed a continuous function $u^\alpha(t, x)$ such that

$$u^\alpha(t, x) = U^\circ(t, x), \quad \text{if} \quad \{t, x\} \in N_1, \qquad u^\alpha(t, x) \in U_t, \quad \text{if} \quad \{t, x\} \in N_2$$

Then, by computing in the region $\varepsilon^\circ(t, x, \vartheta_0^M(t_0, x_0)) > 0$ the derivative $d\varepsilon^\circ / dt$ along the motion of system (1.1) generated by the strategy $u^\alpha(t, x)$ and by any admissible realization $v(t)$, we obtain inequality (2.8). By integrating this inequality we convince ourselves that condition (3.3) holds. This proves the theorem.

4. Let us consider Problem 3 under the condition that the constraints on the players' controls are specified by conditions (2.1), (2.10) and that $\mu > \nu$ and the functions $B(t) = C(t) = b(t)$ in Eq. (1.1).

Then the regular case of (3.1) holds and for any $\alpha > 0$ we can indicate a strategy $V^\alpha(t, x)$ ensuring the evasion by the motion $\{x[t]\}_m$ of going onto the set M until the instant $\vartheta_0^M(t_0, x_0) - {}^1\!/_2\alpha$. Following the results in [6], we can determine the strategy $V^\alpha(t, x)$ as follows. We introduce the sets N_1 and N_2:

$$N_1 = \{t, x\colon \min_\vartheta [\varepsilon^\circ(t, x, \vartheta)] > 0 \quad \text{for } t \leqslant \vartheta \leqslant \vartheta_0^M(t_0, x_0) - {}^1\!/_2\alpha\}$$
$$N_2 = \{t, x\colon \min_\vartheta [\varepsilon^\circ(t, x, \vartheta)] \leqslant 0 \quad \text{for } t \leqslant \vartheta \leqslant \vartheta_0^M(t_0, x_0) - {}^1\!/_2\alpha\}$$

Then for each $\{t, x\} \in N_1$ the strategy $V^\alpha(t, x)$ consists of those and only those v° for which the following condition is valid

$$g'(t, x) b(t) v^\circ = \max_v [g'(t, x), b(t) v] \quad \text{for } |v| \leqslant \nu \tag{4.1}$$

$$g'(t, x) = -\operatorname{grad} \lambda(t, x), \qquad \lambda(t, x) = \int_t^\chi \frac{d\xi}{\varepsilon^\circ(t, x, \xi)} \qquad (\chi = \vartheta_0^M - {}^1\!/_2\alpha) \tag{4.2}$$

The set $V^\alpha(t, x)$ coincides with V_t in region N_2. The following approximation theorem holds here just as for Problems 1 and 2.

T h e o r e m 4.1. For any $\alpha > 0$ there exists a continuous strategy $v^\alpha(t, x)$ such that the relation

$$\inf_u \inf_{x[t]} \min_\vartheta \{\rho [\{x(\vartheta)_m\}, M] \mid \chi [u(t), v^\alpha(t, x), t_0, x_0]\} > 0$$
$$t_0 \leqslant \vartheta \leqslant \vartheta_0^M(t_0, x_0) - \alpha \tag{4.3}$$

is valid.

Let us prove the theorem. We introduce the function $h(t, x) = g'(t, x)\, b(t)$. Then formula (4.1) implies that the evasion strategy is determined by the condition

$$V^\alpha(t, x) = \begin{cases} \nu, & \text{if} \quad h(t, x) > 0 \text{ and } \beta(t, x) > 0 \\ -\nu, & \text{if} \quad h(t, x) < 0 \text{ and } \beta(t, x) > 0 \\ -\nu \leqslant |v^\circ| \leqslant \nu, & \text{if} \quad h(t, x) = 0 \text{ or } \beta(t, x) \leqslant 0 \end{cases} \tag{4.4}$$

$$\beta(t, x) = \min_\vartheta \varepsilon^\circ(t, x, \vartheta) \quad \text{for } t \leqslant \vartheta \leqslant \vartheta_0^M(t_0, x_0) - {}^1\!/_2\alpha$$

We remark that when $\varepsilon^\circ(t, x, \vartheta) > 0$ the function $\varepsilon^\circ(t, x, \vartheta)$ satisfies a Lipschitz condition in ϑ

$$|\varepsilon^\circ(t, x, \vartheta_1) - \varepsilon^\circ(t, x, \vartheta_2)| \leqslant L\, |\vartheta_1 - \vartheta_2|, \qquad L > 0$$

We introduce the sets

$$N_1 = \{t,\ x:\quad h(t,x) > \nu'(k);\ \beta(t,x) > 0\},\quad N_4 = \{t,\ x:\quad \beta(t,x) > \delta\}$$
$$N_2 = \{t,\ x:\ h(t,x) < = \nu'(k);\ \beta(t,x) > 0\},\quad N_5 = \{t,\ x:\ 0 < \beta(t,x) \leqslant \delta\}$$
$$N_3 = \{t,\ x:\ |\,h(t,x)\,| < \nu'(k);\ \beta(t,x) > 0\},\quad N_6 = \{t,\ x:\quad \beta(t,x) \leqslant 0\}$$
$$\nu'(k) = k/2\nu\,[\vartheta_0^M(t_0,\ x_0) - t_0]$$

Here k and δ are positive numbers satisfying the condition

$$k + \lambda(t_0,\ x_0) < \frac{1}{L}\ln\frac{L\alpha}{2\delta} \tag{4.5}$$

We now define a continuous strategy $v^\alpha(t,\ x)$ by setting

$$v^\alpha(t,x) = \begin{cases} v(t,\ x) & \{t,\ x\} \in N_4, \\ v^*(t,\ x), & \{t,\ x\} \in N_5, \\ 0, & \{t,\ x\} \in N_6, \end{cases} \tag{4.6}$$

$$v^*(t,\ x) = \delta^{-1}v(t,\ x)\beta(t,\ x)$$

Here the continuous function $v(t,\ x)$ is given by the relation

$$v(t,\ x) = \begin{cases} \nu, & \{t,\ x\} \in N_1 \\ -\nu, & \{t,\ x\} \in N_2 \\ \nu^*, & \{t,\ x\} \in N_3 \end{cases}$$

where

$$\nu^* = 2\nu^2 k^{-1}(\vartheta_0^M(t_0,\ x_0), - t_0),\ h(t,\ x)$$

Let us show that the strategy $v^\alpha(t,\ x)$ thus defined ensures that the motion $\{x[t]\}_m$ will evade going onto the set M until the instant $\vartheta_0^M(t_0,\ x_0) - \alpha$. Obviously, to do this it is enough to show that the inequality $\beta[t] > \delta$ holds upto the instant $\vartheta_0^M(t_0,\ x_0) - \alpha$ along the motion of system (1.1), generated by the controls $u(t)$, $v^\alpha(t,\ x)$. Indeed, the condition $\beta(t_0,\ x_0) = \beta[t_0] > \delta$ is fulfilled at the initial instant t_0. Let us assume the contrary. Suppose that the equality $\beta[t_1]=\delta$ is fulfilled for the first time at the instant t_1. This means that the vector $\{t_1,\ x[t_1]\}$ hits onto the boundary of set N_5. Then, according to [6], by computing the derivative $d\lambda/dt$ along the motion of system (1.1) generated by the controls $u(t)$, $v^\alpha(t,\ x)$ and by taking into account that $v^\alpha(t,\ x)$ differs from $V^\alpha(t,\ x)$ when $\{t,\ x\} \in N_4$, only on set N_3, we obtain

$$\frac{d\lambda}{dt} \leqslant \frac{k}{\vartheta_0^M(t_0,\ x_0) - t_0} \tag{4.7}$$

It follows that

$$\lambda[t_1] \leqslant \lambda[t_0] + k \tag{4.8}$$

On the other hand, for all values of $\{t,\ x\}$ such that $\beta(t,\ x) = \delta$ and $t \leqslant \vartheta_0^M(t_0,\ x_0)$ the inequality

$$\lambda(t,\ x) > \frac{1}{L}\ln\frac{L\alpha}{2\delta}$$

is valid and, hence,

$$\lambda(t_1,\ x[t_1]) = \lambda[t_1] > \frac{1}{L}\ln\frac{L\alpha}{2\delta} \tag{4.9}$$

Then from (4.8), (4.9) it follows

$$\frac{1}{L}\ln\frac{L\alpha}{2\delta} < \lambda[t_0] + k \tag{4.10}$$

Conditions (4.5) and (4.10) contradict each other and, hence, the motion $\{x[t]\}_m$ does not leave the set N_4 until the instant $\vartheta_0^M(t_0,\ x_0) - \alpha$. This proves the theorem.

5. In conclusion we consider a more general case when the constraints on the controls

u and v are given in form (1.2), but U_t and V_t are arbitrary closed bounded convex sets continuous in t. Moreover, we shall assume that in Problem 3 the sets U_t and V_t are equally oriented and are similar with a similarity coefficient $\beta > 1$ and that $B(t) = C(t)$.

Lemma 5.1. Suppose that we are given a continuous r-dimensional vector-valued function $h(t, x)$ in an $(n+1)$-dimensional space X_t and let W_t be a closed bounded convex set in an r-dimensional space E_r, containing 0 and continuous in t. Then for any $a > 0$ there exists a continuous function $w^\alpha(t, x) \in W_t$ such that

$$h'(t, x) w^\alpha(t, x) \geqslant \max_w [h'(t, x) w] - \alpha \quad \text{for } w \in W_t \tag{5.1}$$

Lemma 5.1 can be proved in the following way. For any t it is necessary to construct, in the r-dimensional space E_r, a strictly convex set G_t containing W_t such that for any vector $g \in G_t$ we can find a vector $w \in W_t$ such that $\| g - w \| \leqslant \eta$, where $\eta > 0$ is a preassigned arbitrarily small number not dependent on t.

Because the set G_t is strictly convex the maximum of the expression

$$h'(t, x) g^\circ = \max_g [h'(t, x) g] \quad \text{for } g \in G_t \tag{5.2}$$

is attained on a single vector $g^\circ = g^\circ(t, x)$, and we can show that $g^\circ(t, x)$ is a continuous function of $\{t, x\}$ on the set $N = \{t, x\colon h(t, x) \neq 0\}$. If we define a function $w^\circ(t, x) \in W_t$ from the condition

$$\| w^\circ(t, x) - g^\circ(t, x) = \| \min_w \| w - g^\circ(t, x) \|, \quad \text{for } w \in W_t \tag{5.3}$$

then it turns out that it is single-valued since the minimum of expression (5.3) is achieved on the single vector $w^\circ = w^\circ(t, x)$, and is continuous on set N. Furthermore, for any $\alpha > 0$ we can find $\eta > 0$ such that inequality (5.1) holds for $\{t, x\} \in N$. Having defined a function $w^\alpha(t, x)$ by the relations

$$w^\alpha(t, x) = \begin{cases} w^\circ(t, x), & \text{if } \| h(t, x) \| > k^{-1} \quad (5.4) \\ k \| h(t, x) \| w^\circ(t, x), & \text{if } \| h(t, x) \| < k^{-1} \quad (5.5) \end{cases}$$

$$k = \delta / \alpha, \quad \delta = \max_t \max_w \| w \| \quad \text{for } w \in W_t$$

we obtain the assertion of Lemma 5.1.

Using Lemma 5.1 we can construct, for Problems 1–3, continuous approximate strategies $U^\circ(t, x)$, $V^\circ(t, x)$ such that in the regular case Theorems 2.1 – 4.1 will hold not only for constraints of the form (2.1), (2.10) but also in the general case of constraints (1.2). It is required only that we repeat, with conceptual variations, those approximate constructions which we have described above in the scalar case.

Note 5.1. In many particular cases the approximations dealt with in Lemma 5.1 can be written out in explicit form without great difficulty. For example, if the constraints on the controls u and v have the form

$$U_t = \{u_1, \ldots, u_r\colon \ | u_i | \leqslant \mu_i(t)\}, \qquad V_t = \{v_1, \ldots, v_s \ | v_j | > \nu_j(t)\} \tag{5.6}$$
$$\mu_i(t) > 0 \quad (i = 1, \ldots, r), \qquad \nu_j(t) > 0 \quad (j = 1, \ldots, s)$$

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